W4L7 - TRANSLATION THEOREMS AND DERIVATIVES OF LAPLACE TRANSFORMS

Given $\mathcal{L}[f(t)] = F(s)$ we can find $\mathcal{L}[e^{at} f(t)]$ by translating $F(s)$ to $F(s-a)$

FIRST TRANSLATION THEOREM:

If a is any real number, then $\mathcal{L}[e^{at} f(t)] = F(s-a)$ where $F(s) = \mathcal{L}[f(t)]$

Proof: $$\mathcal{L}[e^{at} f(t)] = \int_0^\infty e^{-st} e^{at} f(t)\,dt$$
$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

by def, $= F(s-a)$ ■

EX: $$\mathcal{L}[e^{4t} t^2] = \left.\frac{2!}{s^3}\right|_{s \to s-4} = \boxed{\frac{2!}{(s-4)^3}}$$

EX: $$\mathcal{L}[e^{-\pi t} \sin 5t] = \boxed{\frac{5}{(s+\pi)^2 + 25}}$$

INVERSE FORM OF 1st TRANSLATION THEOREM:

$$\mathcal{L}^{-1}[F(s-a)] = e^{at} f(t)$$

EX: $$\mathcal{L}^{-1}\left[\frac{s}{s^2+6s+11}\right]$$

Complete the square

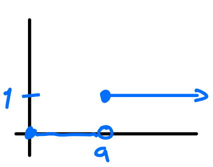
$$\frac{s}{s^2+6s+11} = \frac{s}{(s+3)^2+2} = \frac{s+3}{(s+3)^2+2} - \frac{3}{(s+3)^2+2}$$

$$\mathcal{L}^{-1}\left[\frac{s+3}{(s+3)^2+2}\right] + \mathcal{L}^{-1}\left[\frac{-3}{(s+3)^2+2}\right]$$

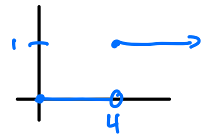
$$= \boxed{e^{-3t} \cos \sqrt{2}t - \frac{3}{\sqrt{2}} e^{-3t} \sin \sqrt{2}t}$$

UNIT STEP FUNCTION:

$$\mathcal{U}(t-a) = \begin{cases} 0 & 0 \le t \le a \\ 1 & t \ge a \end{cases}$$

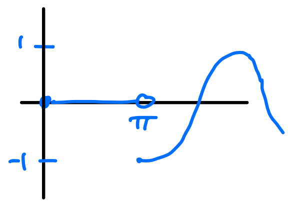


EX: $\mathcal{U}(t-4)$



When a function is multiplied by $\mathcal{U}$, it turns off part of the graph

EX: $f(t) = \cos t \, \mathcal{U}(t-\pi)$



OTHER NOTATIONS:

$$f(t) = \begin{cases} g(t) & 0 \le t \le a \\ h(t) & t \ge a \end{cases}$$

compact form:

$$f(t) = g(t) - g(t)\,\mathcal{U}(t-a) + h(t)\,\mathcal{U}(t-a)$$

Verify: $\begin{cases} g(t) - g(t)(0) + h(t)\,0 & 0 \le t \le a \\ g(t) - g(t)(1) + h(t)(1) & t \ge a \end{cases}$

If:

$$f(t) = \begin{cases} 0 & 0 \le t < a \\ g(t) & a \le t < b \\ 0 & t \ge b \end{cases}$$

compact form:

$$f(t) = g(t)\,\mathcal{U}(t-a) - g(t)\,\mathcal{U}(t-b)$$

EX: $E(t) = \begin{cases} 20t & 0 \le t < 5 \\ 0 & t \ge 5 \end{cases}$

$$\boxed{E(t) = 20t - 20t\,\mathcal{U}(t-5)} + (0\,\mathcal{U}(t-5))$$